

Indecomposable 1-morphisms of $\dot{\mathcal{U}}_3^+$ and the canonical basis of $U_q^+(\mathfrak{sl}_3)$

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January 22, 2013

Abstract

We compute the indecomposable objects of $\dot{\mathcal{U}}_3^+$ – the category that categorifies the positive half of the quantum \mathfrak{sl}_3 , and we decompose an arbitrary object into indecomposable ones. On decategorified level we obtain the Lusztig’s canonical basis of the positive half $U_q^+(\mathfrak{sl}_3)$ of the quantum \mathfrak{sl}_3 . We also categorify the higher quantum Serre relations in $U_q^+(\mathfrak{sl}_3)$, by defining a certain complex in the homotopy category of $\dot{\mathcal{U}}_3^+$ that is homotopic to zero. We work with the category $\dot{\mathcal{U}}_3^+$ that is defined over the ring of integers. This paper is based on the (extended) diagrammatic calculus introduced to categorify quantum groups.

1 Introduction

In recent years there has been a lot of work on a diagrammatic categorification of quantum groups, initiated by Lauda’s diagrammatic categorification [7] of the Lusztig’s idempotented version of $\dot{U}_q(\mathfrak{sl}_2)$. This was extended by Khovanov and Lauda in [5] to $\dot{U}_q(\mathfrak{sl}_n)$ and also in [4] to the positive half of an arbitrary quantum group $U_q^+(\mathfrak{g})$. In [12, 13], Webster modified the construction, and obtained a categorification of an arbitrary quantum group.

General framework of these constructions is to define a certain 2-category \mathcal{U} whose 1-morphisms categorify the generators of a quantum group, and whose 2-morphisms are \mathbb{K} -linear combinations of certain planar diagrams modulo local relations, with \mathbb{K} being a field. Then a 2-category $\dot{\mathcal{U}}$ is defined as the Karoubi envelope of the 2-category \mathcal{U} , i.e. as the smallest 2-category containing \mathcal{U} in which all idempotents (2-morphisms) split. Finally, it is shown that the split Grothendieck group of $\dot{\mathcal{U}}$ is isomorphic to the corresponding quantum group.

In the case of the categorification of the positive half of quantum groups the 2-categories \mathcal{U} and $\dot{\mathcal{U}}$ have a single object. Thus one can see them as monoidal 1-categories. Since in this paper we are interested in categorifications of positive halves of quantum groups, we shall always assume that \mathcal{U}

and $\dot{\mathcal{U}}$ are monoidal (1-)categories.

In order to be able to use diagrammatical calculus in $\dot{\mathcal{U}}$ directly (and not just in \mathcal{U}), in [6] the extension of the calculus – so called thick calculus – was introduced in the case of quantum \mathfrak{sl}_2 . The lines labelled a correspond to objects of $\dot{\mathcal{U}}$ that categorify the divided powers $E^{(a)}$ of the generators of the quantum group. The consequence of [6], and of the thick calculus, is that now one can take \mathbb{Z} -linear combinations of planar diagrams as morphisms of \mathcal{U} .

In this paper, we use thick calculus to study the properties of the category $\dot{\mathcal{U}}_n^+$ that categorifies the positive half of the quantum \mathfrak{sl}_n . The thick calculus can be extended directly to include the categorification of \mathfrak{sl}_n (see [10]). In particular, the category $\dot{\mathcal{U}}_n^+$ is defined over the ring of integers.

In Section 4 of this paper, we compute the indecomposable objects of $\dot{\mathcal{U}}_3^+$. More precisely, in Theorem 2, we show that these are the following:

$$\mathcal{B} = \left\{ \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)} \{t\}, \mathcal{E}_2^{(a)} \mathcal{E}_1^{(b)} \mathcal{E}_2^{(c)} \{t\} \mid b \geq a + c, \quad a, c \geq 0, \quad t \in \mathbb{Z} \right\},$$

with $\mathcal{E}_1^{(a)} \mathcal{E}_2^{(a+c)} \mathcal{E}_1^{(c)} \{t\} \cong \mathcal{E}_2^{(c)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(a)} \{t\}$, for $a, c \geq 0, t \in \mathbb{Z}$. Moreover, we prove that an arbitrary object of $\dot{\mathcal{U}}_3^+$ can be decomposed as a direct sum of the elements of \mathcal{B} .

The main result (Theorem 3), is the categorification of the $U_q^+(\mathfrak{sl}_3)$ relation:

$$E_1^{(a)} E_2^{(b)} E_1^{(c)} = \sum_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_2^{(p)} E_1^{(a+c)} E_2^{(r)}, \quad \text{for } b \leq a+c. \quad (1)$$

By decategorifying the set of indecomposables from \mathcal{B} with no shifts, we obtain the set

$$B = \left\{ E_1^{(a)} E_2^{(b)} E_1^{(c)}, \quad E_2^{(a)} E_1^{(b)} E_2^{(c)} \mid b \geq a + c, \quad a, c \geq 0 \right\},$$

with $E_1^{(a)} E_2^{(a+c)} E_1^{(c)} = E_2^{(c)} E_1^{(a+c)} E_2^{(a)}$, for $a, c \geq 0$. The set B is the Lusztig's canonical basis of $U_q^+(\mathfrak{sl}_3)$ (see [9]), and one of its remarkable properties is that its structure constants are from $\mathbb{N}[q, q^{-1}]$.

Thus, in this way, we have proved that the indecomposable objects of $\dot{\mathcal{U}}_3^+$ lift the canonical basis of $U_q^+(\mathfrak{sl}_3)$. We note once again that we are working in the category that is defined over the ring of integers \mathbb{Z} , i.e. the 1-morphisms are \mathbb{Z} -linear combinations of certain planar diagrams.

Previous results on this topic were obtained in the case when the category is defined over a field, i.e. when the 1-morphisms are \mathbb{K} -linear combinations of planar diagrams, for some field \mathbb{K} . The result that the indecomposable objects of that category lift the Lusztig canonical basis in the case of \mathfrak{sl}_3 was obtained by Khovanov and Lauda [3]. Furthermore, Brundan and Kleshchev [1] have extended the result to the case of affine \mathfrak{sl}_n . Finally, for $\mathbb{K} = \mathbb{C}$, Varagnolo and Vasserot [11] proved this fact for any simply-laced \mathfrak{g} .

The other goal of this paper is to categorify the higher quantum Serre relations for E_1 and E_2 . The higher quantum Serre relations for the generators E_1 and E_2 (and also for the generators E_r and E_s of the positive half of an arbitrary quantum group $U_q^+(\mathfrak{g})$ with $r \cdot s = -1$) are

$$\sum_{i=0}^m (-1)^i q^{\pm(m-n-1)i} E_1^{(m-i)} E_2^{(n)} E_1^{(i)} = 0, \text{ for } m > n > 0. \quad (2)$$

The relation (2) can be obtained by summing appropriately some relations of the form (1). Thus from the categorification of (1) (Theorem 3), one can obtain a decomposition that lifts (2).

However, in Section 5, we give a direct and simple categorification of the higher quantum Serre relation in the homotopy category of $\dot{\mathcal{U}}_3^+$ – the category of complexes in $\dot{\mathcal{U}}_3^+$, modulo homotopies.

Since the higher quantum Serre relations have the form of an alternating sum, it is natural to look for a categorification in the form of a complex of objects of $\dot{\mathcal{U}}_3^+$ that lift the summands of (2). In Theorem 6, we define such a complex and show that it is homotopic to zero. Moreover, the differentials and the homotopies have a particularly simple form.

Finally, all results from this paper about the generators E_1 and E_2 , and objects \mathcal{E}_1 and \mathcal{E}_2 , are valid in an arbitrary quantum group $U_q(\mathfrak{g})$ and in the categorification of its positive half, for the generators E_r and E_s , and for \mathcal{E}_r and \mathcal{E}_s , respectively, with r and s satisfying $r \cdot s = -1$.

2 $U_q^+(\mathfrak{sl}_n)$

In this section we give basic definitions of the positive half of quantum \mathfrak{sl}_n – denoted $U_q^+(\mathfrak{sl}_n)$. We also give some of its combinatorial properties in the case $n = 3$. These properties are also valid for any two generators E_i and E_j of an arbitrary quantum group when $i \cdot j = -1$.

Let $n \geq 2$ be fixed. The index set of quantum \mathfrak{sl}_n is $I = \{1, 2, \dots, n-1\}$.

The inner product is defined on $\mathbb{Z}[I]$ by setting that for $i, j \in I$:

$$i \cdot j = \begin{cases} 2, & i = j \\ -1, & |i - j| = 1 \\ 0, & |i - j| \geq 2 \end{cases}$$

$U_q^+(\mathfrak{sl}_n)$ is a $\mathbb{Q}(q)$ -algebra generated by E_1, E_2, \dots, E_{n-1} modulo relations:

$$E_i^2 E_j + E_j E_i^2 = [2] E_i E_j E_i, \quad i \cdot j = -1, \quad (3)$$

$$E_i E_j = E_j E_i, \quad i \cdot j = 0. \quad (4)$$

The divided powers of the generators are defined by

$$E_i^{(a)} := \frac{E_i^a}{[a]!}, \quad a \geq 0, \quad i = 1, \dots, n-1.$$

The divided powers satisfy:

$$E_i^{(a)} E_j^{(b)} = E_j^{(b)} E_i^{(a)}, \quad i \cdot j = 0, \quad (5)$$

$$E_i^{(a)} E_i^{(b)} = \begin{bmatrix} a+b \\ a \end{bmatrix} E_i^{(a+b)}, \quad (6)$$

and the quantum Serre relations

$$E_i^{(2)} E_j + E_j E_i^{(2)} = E_i E_j E_i, \quad i \cdot j = -1. \quad (7)$$

The integral form ${}_{\mathbb{Z}}U_q^+(\mathfrak{sl}_n)$ is the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of $U_q^+(\mathfrak{sl}_n)$ generated by $E_i^{(a)}$, for all $i = 1, \dots, n-1$ and $a \geq 0$.

2.1 Combinatorics of $U_q^+(\mathfrak{sl}_3)$

The higher quantum Serre relations (Chapter 7 of [8]) for E_1 and E_2 are:

$$\sum_{r=0}^m (-1)^r q^{\pm(m-n-1)r} E_1^{(m-r)} E_2^{(n)} E_1^{(r)} = 0, \quad m > n > 0, \quad (8)$$

and analogously with E_1 and E_2 switched. In particular, the quantum Serre relations are obtained for $m = 2$ and $n = 1$.

Proposition 1 ([8], Lemma 42.1.2.(d)) *For any three nonnegative integers a, b, c , with $b \leq a + c$, we have:*

$$E_1^{(a)} E_2^{(b)} E_1^{(c)} = \sum_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_2^{(p)} E_1^{(a+c)} E_2^{(r)}. \quad (9)$$

In particular, for $b = a + c$, we have:

$$E_1^{(a)} E_2^{(a+c)} E_1^{(c)} = E_2^{(c)} E_1^{(a+c)} E_2^{(a)}. \quad (10)$$

Finally, both formulas are valid when E_1 and E_2 interchange places.

We note that the higher quantum Serre relations (8) follow from (9), together with the following well-known relation of quantum binomial coefficients, valid for any nonnegative integer N :

$$\sum_{k=0}^N (-1)^k q^{\pm(N-1)k} \begin{bmatrix} N \\ k \end{bmatrix} = 0. \quad (11)$$

All relations from above for E_1 and E_2 are also valid in an arbitrary quantum group for two generators E_i and E_j , with $i \cdot j = -1$.

2.2 Monomials in $U_q^+(\mathfrak{sl}_3)$

By a monomial, we mean a vector of the form $E_1^{(a_1)} E_2^{(b_2)} E_1^{(a_2)} \dots E_1^{(a_n)} E_2^{(b_n)}$. The number of nonzero exponents a_i and b_i we call the length of the vector. Let B be the following set of monomials:

$$B = \{E_1^{(a)} E_2^{(b)} E_1^{(c)}, E_2^{(a)} E_1^{(b)} E_2^{(c)} \mid b \geq a + c, a, b, c \geq 0\}, \quad (12)$$

where we have $E_1^{(a)} E_2^{(a+c)} E_1^{(c)} = E_2^{(c)} E_1^{(a+c)} E_2^{(a)}$, for all $a, c \geq 0$.

The set B is the Lusztig's canonical basis of $U_q^+(\mathfrak{sl}_3)$, and its structure constants are from $\mathbb{N}[q, q^{-1}]$.

We also have the following:

Theorem 1 *Every monomial from $U_q^+(\mathfrak{sl}_3)$ can be expressed as a linear combination of vectors from B , with coefficients from $\mathbb{N}[q, q^{-1}]$.*

Proof:

By induction on length. By Proposition 1 any monomial of length at most 3 can be expressed as a linear combination of the vectors from B , which proves the base of induction.

Now, suppose that a monomial v has length l , with $l \geq 4$. Then it contains a piece of the form $E_1^{(a)} E_2^{(b)} E_1^{(c)} E_2^{(d)}$, with $a, b, c, d > 0$ (or a piece of the form $E_2^{(a)} E_1^{(b)} E_2^{(c)} E_1^{(d)}$, with $a, b, c, d > 0$, which is done completely analogously). Then, we have that at least one of the inequalities $b < a + c$ or $c < b + d$ is satisfied. Suppose that the first one is satisfied (the second one is done in the same way). Then by Proposition 1 we have

$$E_1^{(a)} E_2^{(b)} E_1^{(c)} E_2^{(d)} = \sum_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_2^{(p)} E_1^{(a+c)} E_2^{(r)} E_2^{(d)} =$$

$$= \sum_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} \begin{bmatrix} r+d \\ r \end{bmatrix} E_2^{(p)} E_1^{(a+c)} E_2^{(r+d)},$$

i.e. v can be written as a linear combination of the monomials of length at most $l - 1$, which proves the first part of the theorem.

As for the coefficients, they are all sums and products of the quantum binomial coefficients, and so they are from $\mathbb{N}[q, q^{-1}]$, as wanted. \blacksquare

3 Category \mathcal{U}_n^+

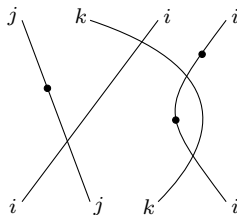
A categorification of the positive half of quantum \mathfrak{sl}_n (and also of an arbitrary quantum group $U_q^+(\mathfrak{g})$) was defined in [4]. The categorification there is given by means of the category of projective modules over certain (diagrammatically defined) rings $R(\nu)$ (see [3]). In this paper we will use a categorification in terms of diagrammatic category \mathcal{U}_n^+ , which is in the spirit of the categorification of the whole quantum \mathfrak{sl}_n (see [5]).

Before going to the definition of \mathcal{U}_n^+ , first we recall some notation and explain the diagrams that appear in its definition (see also [5]).

Let $n \geq 2$ be fixed. By Seq we denote the set of all sequence of finite length of the numbers from $\{1, \dots, n-1\}$. The entries of the elements of Seq we call *colors*. For $\nu = (\nu_1, \dots, \nu_{n-1}) \in \mathbb{N}^{n-1}$, by $\text{Seq}(\nu)$ we denote the set of all sequences $\underline{i} = (i_1, \dots, i_k) \in \text{Seq}$ such that $\#\{j | i_j = l\} = \nu_l$, for all $l = 1, \dots, n-1$. Note that, in particular, we have that $k = \sum_l \nu_l$.

We will use the following diagrammatic calculus of planar diagrams: We consider collections of arcs on the plane connecting k points on one horizontal line with k points on another horizontal line. The positions of k points on the horizontal line are always the points $\{1, \dots, k\} \in \mathbb{R}$. Each arc is labelled by a number from the set $\{1, \dots, n-1\}$ (called the *color* of an arc). We require that arcs have no critical points when projected to y -axis. Also, arcs can intersect, but no triple intersections are allowed. Finally, an arc can carry dots.

The following is an example of a planar diagram:



We identify two planar diagrams if there exists a planar isotopy between them, that does not change the combinatorial type of the diagram and do not create critical points for the projection onto the y -axis.

Note that since we are not allowing the arcs to have critical points when projected to y -axis, we can assume that they are always oriented upwards. In particular, we can see a planar diagram as going from the sequence corresponding to the colors of the bottom end of the strands going to the sequence corresponding to the colors of the top end of the strands (we read the ends of the strands from left to right). For a diagram D , we denote the sequence that corresponds to the bottom (top) end of the strands by $\text{Bot}(D)$ ($\text{Top}(D)$, respectively). In the example above, we have $\text{Bot}(D) = (i, j, k, i)$ and $\text{Top}(D) = (j, k, i, i)$.

Each diagram has a degree, by setting that the degree of a dot is equal to 2, while the degree of a crossing between two arcs that are colored by colors i and j is equal to $-i \cdot j$. In other words, for $i = j$ the degree of a crossing is equal to -2 , for $|i - j| = 1$ (adjacent colors) the degree of a crossing is equal to 1, while for $|i - j| \geq 2$ (distant colors) the degree of a crossing is equal to 0:

$$\text{degree:} \quad \begin{array}{c} | \\ \bullet \\ | \\ i \end{array} \quad +2, \quad \begin{array}{c} \diagup \quad \diagdown \\ i \quad j \end{array} \quad -i \cdot j$$

We also use the following shorthand notation for the sequence of d dots on an edge of a strand:

$$\begin{array}{c} | \\ d \\ | \\ i \end{array} := \left. \begin{array}{c} | \\ \bullet \\ \vdots \\ \bullet \\ | \\ i \end{array} \right\} d$$

3.1 Category \mathcal{U}_n^+

We define category \mathcal{U}_n^+ as follows:

\mathcal{U}_n^+ is the monoidal \mathbb{Z} -linear additive category whose objects and morphisms are the following:

- objects: for each $\underline{i} = (i_1, \dots, i_k) \in \text{Seq}$ and $t \in \mathbb{Z}$, we define $\mathcal{E}_{\underline{i}}\{t\} := \mathcal{E}_{i_1} \dots \mathcal{E}_{i_k}\{t\}$. An object of \mathcal{U}_n^+ is a formal finite direct sum of objects $\mathcal{E}_{\underline{i}}\{t\}$, with $\underline{i} \in \text{Seq}$ and $t \in \mathbb{Z}$.

- morphisms: for $\underline{i} = (i_1, \dots, i_k) \in \text{Seq}(\nu)$ and $\underline{j} = (j_1, \dots, j_l) \in \text{Seq}(\mu)$ the set $\text{Hom}(\mathcal{E}_{\underline{i}}\{t\}, \mathcal{E}_{\underline{j}}\{t'\})$ is empty, unless $\nu = \mu$. If $\nu = \mu$ (and consequently $k = l$), the morphisms from $\mathcal{E}_{\underline{i}}\{t\}$ to $\mathcal{E}_{\underline{j}}\{t'\}$ consists of finite \mathbb{Z} -linear

combinations of planar diagrams going from \underline{i} to \underline{j} , of degree $t - t'$, modulo the following set of local relations (note that all relations preserve the degree):

$$\begin{aligned}
& \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} - \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} = \begin{array}{c} | \\ | \end{array} \quad \begin{array}{c} | \\ | \end{array} \\
& \begin{array}{c} \text{Diagram 9} \end{array} = 0, \quad \begin{array}{c} \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \end{array} \\
& \begin{array}{c} \text{Diagram 12} \end{array} = \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array}, \quad \text{when } i \cdot j = -1 \\
& \begin{array}{c} \text{Diagram 13} \end{array} = \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array}, \quad \text{when } i \cdot j = 0 \\
& \begin{array}{c} \text{Diagram 14} \end{array} = \begin{array}{c} \text{Diagram 15} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 16} \end{array} = \begin{array}{c} \text{Diagram 17} \end{array}, \quad \text{when } i \neq j \\
& \begin{array}{c} \text{Diagram 18} \end{array} = \begin{array}{c} \text{Diagram 19} \end{array}, \quad \text{if } i \neq k \text{ or } i \cdot j \neq -1 \\
& \begin{array}{c} \text{Diagram 20} \end{array} = \begin{array}{c} \text{Diagram 21} \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array}, \quad \text{if } i \cdot j = -1
\end{aligned} \tag{13}$$

$$\begin{aligned}
& \begin{array}{c} \text{Diagram 22} \end{array} = \begin{array}{c} \text{Diagram 23} \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array}, \quad \text{if } i \cdot j = -1
\end{aligned} \tag{14}$$

This ends the definition of \mathcal{U}_n^+ .

We have the following relation in \mathcal{U}_n^+ :

Proposition 2 (Dot Migration) [7] *We have*

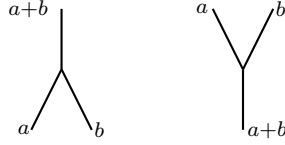
$$\begin{array}{c} \text{Diagram 24} \end{array} - \begin{array}{c} \text{Diagram 25} \end{array} = \begin{array}{c} \text{Diagram 26} \end{array} - \begin{array}{c} \text{Diagram 27} \end{array} = \sum_{r+s=d-1} \begin{array}{c} | \\ | \end{array} + \begin{array}{c} | \\ | \end{array}$$

3.2 Category $\dot{\mathcal{U}}_n^+$ and thick calculus

In [6], the extension of the calculus to thick edges have been introduced. Thick lines categorify the divided powers $E_i^{(a)}$ (see below and Section 4 of [6]).

A thick line is defined in terms of “ordinary” lines from above, and drawn as a strand with an additional label (natural number) a (also called the *thickness* of a strand). In particular, the ordinary strands from above correspond to the case $a = 1$, and are also called *thin* edges or thin strands. For any color the thick edges are defined in the same way as in [6], and we refer the reader to that paper for more details. Here we just recall the basic facts that will be used later on.

Also, now the trivalent vertices in planar diagrams are allowed (called Splitters in [6]), such the sum of the thicknesses of the incoming edges is equal to the sum of the thicknesses of the outgoing edges. Recall that we are assuming that all strands are oriented upwards. The degree of a trivalent vertex (for any color - the labels on the pictures below represent thicknesses)



is equal to $-ab$.

3.2.1 Category $\dot{\mathcal{U}}_n^+$

For a category \mathcal{C} , the Karoubi envelope $Kar(\mathcal{C})$ is the smallest category containing \mathcal{C} , such that all idempotents split (for more details, see e.g. Section 3.4 of [6]).

We define the category $\dot{\mathcal{U}}_n^+$ as the Karoubi envelope of the category \mathcal{U}_n^+ .

Now, in the category $\dot{\mathcal{U}}_n^+$, the planar diagrams with thick edges from above can be interpreted as morphisms whose bottom and top end correspond to certain objects of $\dot{\mathcal{U}}_n^+$. In particular, the object corresponding to bottom (or the top end) of an arc of color i and thickness a is denoted $\mathcal{E}_i^{(a)}$.

As in [6] (see also [10]), the category $\dot{\mathcal{U}}_n^+$ categorifies $U_q^+(\mathfrak{sl}_n)$, in a sense that its split Grothendieck group is isomorphic to the integral form of $U_q^+(\mathfrak{sl}_n)$. The isomorphism sends the class of $\mathcal{E}_i^{(a)}$ to the generator $E_i^{(a)}$ of $U_q^+(\mathfrak{sl}_n)$.

3.2.2 Some properties of thick calculus

Below we give some of the basic properties of thick edges that we shall use in this paper (see [6] for more details). Note that the labels of the strands below denote thickness.

Proposition 3 (Associativity of splitters) *For arbitrary color i (drawn as thick line), we have the following:*

Proposition 4 (Pitchfork lemma) *For any two colors i (drawn as thick line) and j (drawn dashed), we have:*

Proposition 5 (Opening of a Thick Edge) *For any color i (drawn as thick line), we have:*

3.3 Schur polynomials and labels on thick lines

Thick lines are labelled with symmetric polynomials, that correspond to symmetric polynomials in dots on thin edges involved in the definition of a thick line (for precise definition see [6]). In particular, we label thick edges by Schur polynomials, which form the additive basis of the ring of symmetric polynomials.

3.3.1 Schur polynomials

Here we recall briefly the definition and some basic notation and properties of the Schur polynomials. For more details on them see e.g. [6] or [2].

By a partition $\alpha = (\alpha_1, \dots, \alpha_k)$, we mean a non-increasing sequence of non-negative integers. We identify two partitions if they differ by a sequence of zeros at the end. We set $|\alpha| = \sum_i \alpha_i$. If for some a we have $\alpha_{a+1} = 0$, we say that α has at most a parts. We denote the set of all partitions with at most a parts by $P(a)$. Furthermore, by $P(a, b)$ we denote the subset of all partitions α from $P(a)$ such that $\alpha_1 \leq b$.

By $\bar{\alpha}$ we denote the dual (conjugate) partition of α , i.e. $\alpha_j = \#\{i | \alpha_i \geq j\}$. If $\alpha \in P(a, b)$, we define partition $\hat{\alpha}$ by $\hat{\alpha} = (b - \alpha_a, \dots, b - \alpha_1)$. Note that if $\alpha \in P(a, b)$, then $\bar{\alpha} \in P(b, a)$ and $\hat{\alpha} \in P(b, a)$.

For any partition $\alpha \in P(a)$, the Schur polynomial π_α is given by the formula:

$$\pi_\alpha(x_1, x_2, \dots, x_a) = \frac{|x_i^{\alpha_j + a - j}|}{\Delta},$$

where $\Delta = \prod_{1 \leq r < s \leq a} (x_r - x_s)$, and $|x_i^{\alpha_j + a - j}|$ is the determinant of the $a \times a$ matrix whose (i, j) entry is $x_i^{\alpha_j + a - j}$. We let $\pi_\alpha(x_1, x_2, \dots, x_a) = 0$ if some entry of α is negative (α is not a partition then), or if $\alpha_{a+1} > 0$.

For three partitions α , β and γ , the Littlewood-Richardson coefficient $c_{\alpha, \beta}^\gamma$ are given by:

$$\pi_\alpha \pi_\beta = \sum_\gamma c_{\alpha, \beta}^\gamma \pi_\gamma.$$

The coefficients $c_{\alpha, \beta}^\gamma$ are nonnegative integers that can be nonzero only when $|\gamma| = |\alpha| + |\beta|$.

The Littlewood-Richardson coefficients can be naturally extended for more than three partitions: for partitions $\alpha_1, \dots, \alpha_k$ and β , with $k \geq 2$, we define $c_{\alpha_1, \dots, \alpha_k}^\beta$ by:

$$\pi_{\alpha_1} \dots \pi_{\alpha_k} = \sum_\beta c_{\alpha_1, \dots, \alpha_k}^\beta \pi_\beta.$$

For two partitions α and γ , we say that $\alpha \subset \gamma$, if $\alpha_i \leq \gamma_i$ for all $i \geq 1$. If $\alpha \subset \gamma$, we define skew-Schur polynomial $\pi_{\gamma/\alpha}$ by:

$$\pi_{\gamma/\alpha} = \sum_\beta c_{\alpha, \beta}^\gamma \pi_\beta.$$

Finally, the elementary symmetric polynomials $\varepsilon_m(x_1, \dots, x_a)$, for $m = 0, \dots, a$, are special Schur polynomials: $\varepsilon_m(x_1, \dots, x_a) = \pi_{\underbrace{(1, 1, \dots, 1)}_m}((x_1, \dots, x_a))$.

The above lemma implies the following

Lemma 2 *Let $\alpha \in P(a, x)$ and $\beta \in P(b, y)$ be partitions. Then we have that*

$$\begin{array}{c} a+b \\ | \\ \pi_\alpha \quad \pi_\beta \\ \diagup \quad \diagdown \\ a \quad b \\ | \\ a+b \end{array} = s \quad \begin{array}{c} | \\ \pi_\gamma \\ | \\ a+b \end{array}$$

for some partition $\gamma \in P(a+b, x-b+y-a)$ and $s \in \{-1, 0, 1\}$. If $s \neq 0$, then $|\gamma| = |\alpha| + |\beta| - ab$.

Moreover, if $\alpha \in P(a, b)$ and $\beta \in P(b, a)$, then we have:

$$\begin{array}{c} a+b \\ | \\ \pi_\alpha \quad \pi_\beta \\ \diagup \quad \diagdown \\ a \quad b \\ | \\ a+b \end{array} = \delta_{\beta, \hat{\alpha}} (-1)^{|\beta|} \quad \begin{array}{c} | \\ | \\ a+b \end{array}$$

In particular, the left hand side can be nonzero only when $|\alpha| + |\beta| = ab$.

Lemma 3 *Let $\gamma \in P(a)$ and $\psi \in P(a, b)$ be partitions, and let $K = \underbrace{(b, \dots, b)}_a$. Then*

$$\begin{array}{c} a+b \\ | \\ \pi_\psi \quad \pi_\gamma \\ \diagup \quad \diagdown \\ a \quad b \\ | \\ a+b \end{array} = \quad \begin{array}{c} | \\ \pi_{\gamma / (K-\psi)} \\ | \\ a+b \end{array}$$

where $K - \psi = (b - \psi_a, b - \psi_{a-1}, \dots, b - \psi_1)$.

Proof:

$$\begin{array}{c} a+b \\ | \\ \pi_\psi \quad \pi_\gamma \\ \diagup \quad \diagdown \\ a \quad b \\ | \\ a+b \end{array} = \sum_{\mu \in P(a)} c_{\gamma, \psi}^\mu \quad \begin{array}{c} a+b \\ | \\ \pi_\mu \\ \diagup \quad \diagdown \\ a \quad b \\ | \\ a+b \end{array} = \sum_{\nu \in P(a)} c_{\gamma, \psi}^{\nu+K} \quad \begin{array}{c} | \\ \pi_\nu \\ | \\ a+b \end{array}$$

where $\nu + K = (\nu_1 + b, \dots, \nu_a + b)$. Since $\pi_{K/\gamma} = \pi_{K-\gamma}$, we have $c_{\gamma,\psi}^{\nu+K} = c_{\nu,K-\psi}^\gamma$, and so $\pi_{\gamma/(K-\psi)} = \sum_\nu c_{\nu,K-\psi}^\gamma \pi_\nu = \sum_\nu c_{\gamma,\psi}^{\nu+K} \pi_\nu$, which gives the above lemma. \blacksquare

We shall also need the extensions of R2 and R3 like moves for thick edges, in the case when colors i and j of strands involved satisfy $i \cdot j = -1$.

Since in thick calculus each line carries two indices, we shall use the following convention for drawing diagrams, in order to have as small number labels on diagrams as possible.

Notation convention: From now on, for two colors (indexes) that satisfy $i \cdot j = -1$, we shall draw lines corresponding to i as normal (straight) lines, while the lines corresponding to color j , we shall draw as curly lines:

$$\mathcal{E}_i^{(a)} : \begin{array}{c} \text{---} \text{---} \text{---} \\ | \\ a \end{array} \quad \mathcal{E}_j^{(b)} : \begin{array}{c} \text{---} \text{---} \text{---} \\ \} \\ b \end{array}$$

Thus, from now on, each line carries one label, and that label represents its thickness.

The two proposition below are straightforward extensions of the thin R2 and R3 relations (13) and (14) (see [6] and [10] for more details).

Proposition 6 (Thick R2 Move) *We have*

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ a \quad b \end{array} = \sum_{\alpha \in P(a,b)} \begin{array}{c} \text{---} \text{---} \text{---} \\ \bullet \\ a \end{array} \begin{array}{c} \text{---} \text{---} \text{---} \\ \bullet \\ b \end{array}$$

Proposition 7 (Thick R3 Move) *We have*

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ a \quad c \quad b \end{array} = \sum_{i=0}^{\min(a,b,c)} \sum_{\alpha,\beta,\gamma \in P(i,c-i)} c_{\alpha\beta\gamma}^{K_i} \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ a \quad c \quad b \end{array}$$

where $K_i = \underbrace{(c-i, c-i, \dots, c-i)}_i$, for $i > 0$, and $K_0 = 0$.

4 Indecomposables in $\dot{\mathcal{U}}_3^+$

In this section we compute the indecomposable objects of $\dot{\mathcal{U}}_3^+$. We show that these are exactly the objects that categorify the elements of the canonical basis set B . Furthermore, we decompose an arbitrary object as a direct sum of these indecomposables, by categorifying (9).

Theorem 2 *The objects $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}$ and $\mathcal{E}_2^{(a)}\mathcal{E}_1^{(b)}\mathcal{E}_2^{(c)}\{t\}$, for nonnegative a, b, c and $t \in \mathbb{Z}$, with $b \geq a + c$, are indecomposable in $\dot{\mathcal{U}}_3^+$. Neither two of them are isomorphic, except $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)}\{t\} \cong \mathcal{E}_2^{(c)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(a)}\{t\}$, for $a, c \geq 0$, $t \in \mathbb{Z}$.*

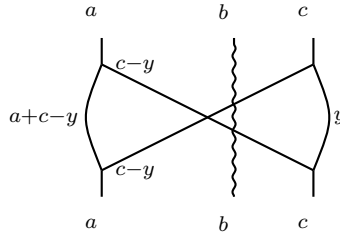
Proof:

First of all, note that it is enough to prove the claim for the objects with no shifts. We will show that $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}$, with $b \geq a + c$, are indecomposable by showing that the graded rank of their endomorphism rings satisfy:

$$\mathrm{rk}_q \mathrm{Hom}_{\dot{\mathcal{U}}^*}(\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}, \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}) \in 1 + q\mathbb{N}[q]. \quad (15)$$

Here $\dot{\mathcal{U}}^*$ is the category with the same objects as $\dot{\mathcal{U}}_3^+$, while morphism between two objects can have arbitrary degree. More precisely, for any two objects $x, y \in \dot{\mathcal{U}}^*$, we have $\mathrm{Hom}_{\dot{\mathcal{U}}^*}(x, y) = \oplus_{s \in \mathbb{Z}} \mathrm{Hom}_{\dot{\mathcal{U}}_3^+}(x\{s\}, y)$ (see e.g. [7]).

Any diagram from $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}$ to itself can be represented in a “standard” form. Namely, we split thick edges into thin ones, and it is well-known (see e.g. [7],[5]) that any such diagram can be reduced to a diagram where any two thin lines intersect at most once, and all dots from one line are only at one segment. By regrouping the thin edges back into the thick ones, in our case, the only nonzero diagrams are of the form



possibly with some dots on it. Thus, the lowest degree diagrams are the dotless ones of the form above, where $y \leq c$ varies.

Now, for $y = c$ we have the identity - the degree zero map. If $y < c$, the degree of this dotless diagram is

$$-2(c-y)(a-c+y) - 2(c-y)y - 2(c-y)^2 + 2b(c-y) = 2(c-y)(b-a-y) > 0,$$

for $b \geq a + c$, and so we have proved (15). Thus, we have that $\mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}$ is indecomposable for $b \geq a + c$, as wanted. In the same way, we have that $\mathcal{E}_2^{(a)} \mathcal{E}_1^{(b)} \mathcal{E}_2^{(c)}$ is indecomposable for $b \geq a + c$, as well.

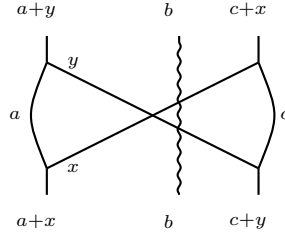
In order to prove that two indecomposables of the form $\mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}$ and $\mathcal{E}_1^{(p)} \mathcal{E}_2^{(k)} \mathcal{E}_1^{(r)}$, with $(a, b, c) \neq (p, k, r)$ are not isomorphic, we will show that

$$\mathrm{rk}_q \mathrm{Hom}_{\mathcal{U}^*}(\mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}, \mathcal{E}_1^{(p)} \mathcal{E}_2^{(k)} \mathcal{E}_1^{(r)}) \in q\mathbb{N}[q].$$

This Hom-space can be nonempty only when $k = b$ and $p + r = a + c$, and so we are left with proving:

$$\mathrm{rk}_q \mathrm{Hom}_{\mathcal{U}^*}(\mathcal{E}_1^{(a+x)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c+y)}, \mathcal{E}_1^{(a+y)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c+x)}) \in q\mathbb{N}[q], \quad (16)$$

when at least one of x and y is nonzero, with $b \geq a + c + x + y$. Again, a general (dotless) diagram from the last Hom-space has the following form



The degree of such dotless diagram is equal to

$$\begin{aligned} & -ax - ay - cx - cy - 2xy + b(x + y) \geq \\ & \geq -(a + c)(x + y) - 2xy + (a + c)(x + y) + (x + y)^2 = x^2 + y^2 > 0, \end{aligned}$$

as wanted.

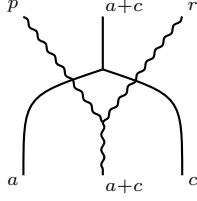
Finally, to prove that two indecomposables of the form $\mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}$ and $\mathcal{E}_2^{(p)} \mathcal{E}_1^{(k)} \mathcal{E}_2^{(r)}$ are not isomorphic, again we compute

$$\mathrm{rk}_q \mathrm{Hom}_{\mathcal{U}^*}(\mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}, \mathcal{E}_2^{(p)} \mathcal{E}_1^{(k)} \mathcal{E}_2^{(r)}),$$

for $b \geq a + c$ and $k \geq p + r$. In order this Hom-space to be non-empty, we must have $b = p + r$ and $k = a + c$, and so $b = k = p + r = a + c$. Hence, we are left with computing

$$\mathrm{rk}_q \mathrm{Hom}_{\mathcal{U}^*}(\mathcal{E}_1^{(a)} \mathcal{E}_2^{(a+c)} \mathcal{E}_1^{(c)}, \mathcal{E}_2^{(p)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(r)}), \quad (17)$$

with $p + r = a + c$. Again, a general dotless morphism from this Hom-space has the following form



The degree of the diagram from above is equal to

$$-ac - pr + ap + cr = (a - r)(p - c) = (p - c)^2 > 0,$$

for $p \neq c$. Thus, for $p \neq c$, the graded rank (17) is from $q\mathbb{N}[q]$, and so we have that the two indecomposable objects $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}$ and $\mathcal{E}_2^{(p)}\mathcal{E}_1^{(k)}\mathcal{E}_2^{(r)}$ are not isomorphic.

The only possibility left for the isomorphism is between $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)}$ and $\mathcal{E}_2^{(c)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(a)}$, and below (Theorem 3) we show that they are indeed isomorphic. \blacksquare

4.1 Decomposition of an arbitrary object of $\dot{\mathcal{U}}_3^+$

In this section we decompose an arbitrary object from $\dot{\mathcal{U}}_3^+$ as a direct sum of the indecomposable ones from the previous section. We do this by categorifying the Proposition 1, i.e. by proving the following

Theorem 3 *For $b \leq a + c$, we have the following canonical decomposition*

$$\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)} \cong \bigoplus_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \bigoplus_{\alpha \in P(c-p, a-r)} \mathcal{E}_2^{(p)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(r)} \{2|\alpha| - (c-p)(a-r)\}. \quad (18)$$

Note that this indeed categorifies (9), since

$$\begin{bmatrix} a+c-b \\ c-p \end{bmatrix} = \begin{bmatrix} a-r+c-p \\ c-p \end{bmatrix} = \sum_{\alpha \in P(c-p, a-r)} q^{2|\alpha| - (c-p)(a-r)}.$$

Remark 1 *The decomposition (18) is also valid in general, i.e. if we replace \mathcal{E}_1 and \mathcal{E}_2 by \mathcal{E}_r and \mathcal{E}_s , respectively, with $r \cdot s = -1$.*

Also, in the same way as in the Theorem 1 (by changing sums with direct sums), the decomposition (18), together with the decomposition (see [[6], Theorem 5.1])

$$\mathcal{E}_i^{(a)}\mathcal{E}_i^{(b)} \cong \bigoplus_{\alpha \in P(a,b)} \mathcal{E}_i^{(a+b)} \{2|\alpha| - ab\},$$

that categorifies (6), implies the decomposition of an arbitrary object as a direct sum of the indecomposable ones from the previous section. Thus, the decomposition (18) gives:

Theorem 4 *The set of indecomposable objects of $\dot{\mathcal{U}}_3^+$ is the following set \mathcal{B} :*

$$\mathcal{B} = \{\mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)} \{t\}, \mathcal{E}_2^{(a)} \mathcal{E}_1^{(b)} \mathcal{E}_2^{(c)} \{t\} \mid b \geq a + c, a, b, c \geq 0, t \in \mathbb{Z}\}. \quad (19)$$

No two elements from \mathcal{B} are isomorphic, except

$$\mathcal{E}_1^{(a)} \mathcal{E}_2^{(a+c)} \mathcal{E}_1^{(c)} \{t\} \cong \mathcal{E}_2^{(c)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(a)} \{t\}, \quad a, c \geq 0, t \in \mathbb{Z}.$$

An arbitrary object of $\dot{\mathcal{U}}_3^+$ can be decomposed as a direct sum of the elements from \mathcal{B} .

In this way we have obtained a bijection between the canonical basis B of $U_q^+(\mathfrak{sl}_3)$ and the indecomposable objects from \mathcal{B} with no shifts, given by:

$$\begin{aligned} E_1^{(a)} E_2^{(b)} E_1^{(c)} &\mapsto \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}, \quad b \geq a + c, \\ E_2^{(a)} E_1^{(b)} E_2^{(c)} &\mapsto \mathcal{E}_2^{(a)} \mathcal{E}_1^{(b)} \mathcal{E}_2^{(c)}, \quad b \geq a + c, \\ E_1^{(a)} E_2^{(a+c)} E_1^{(c)} = E_2^{(c)} E_1^{(a+c)} E_2^{(a)} &\mapsto \mathcal{E}_1^{(a)} \mathcal{E}_2^{(a+c)} \mathcal{E}_1^{(c)} \cong \mathcal{E}_2^{(c)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(a)}. \end{aligned}$$

So, we are left with proving the decomposition (18) (i.e. Theorem 3) and that is done in the rest of this section.

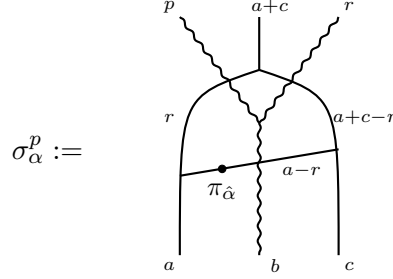
4.2 Proof of Theorem 3

First of all, the conditions $p + r = b$, $p \leq c$ and $r \leq a$ together, are equivalent to $\max(0, b - a) \leq p \leq \min(b, c)$, with $r = b - p$. Then, for every non-negative integer p with $\max(0, b - a) \leq p \leq \min(b, c)$, and partition $\alpha \in P(c - p, a - r)$, where $r = b - p$, we define the following 2-morphisms:

$$\lambda_\alpha^p : \mathcal{E}_2^{(p)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(r)} \{2|\alpha| - (c - p)(a - r)\} \longrightarrow \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}, \quad (20)$$

$$\lambda_\alpha^p := (-1)^{r(a+c-r)+|\alpha|}$$

$$\sigma_\alpha^p : \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)} \longrightarrow \mathcal{E}_2^{(p)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(r)} \{2|\alpha| - (c-p)(a-r)\}, \quad (21)$$



$$e_\alpha^p := \lambda_\alpha^p \sigma_\alpha^p : \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)} \longrightarrow \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(c)}. \quad (22)$$

The following lemma is the key result:

Lemma 4 *Let $b \leq a + c$. Let $\max(0, b - a) \leq p, p' \leq \min(b, c)$, $\alpha \in P(c - p, a - r)$ and $\alpha' \in P(c - p', a - r')$, where $r = b - p$ and $r' = b - p'$. Then:*

$$\sigma_{\alpha'}^{p'} \lambda_\alpha^p = \delta_{p,p'} \delta_{\alpha,\alpha'} \text{Id}_{\mathcal{E}_2^{(p)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(r)}}. \quad (23)$$

The last lemma implies the theorem below, which in turn proves the wanted decomposition (18).

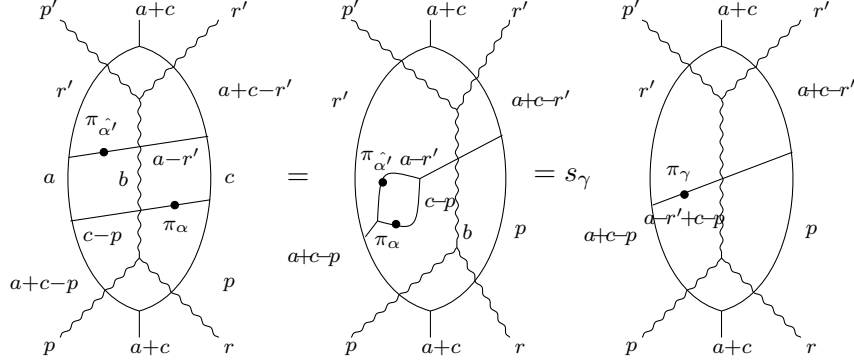
Theorem 5 *The collection $\{e_\alpha^p\}$ is a collection of mutually orthogonal idempotents.*

Proof of Lemma 4:

In pictures, the statement of the lemma above is the following:

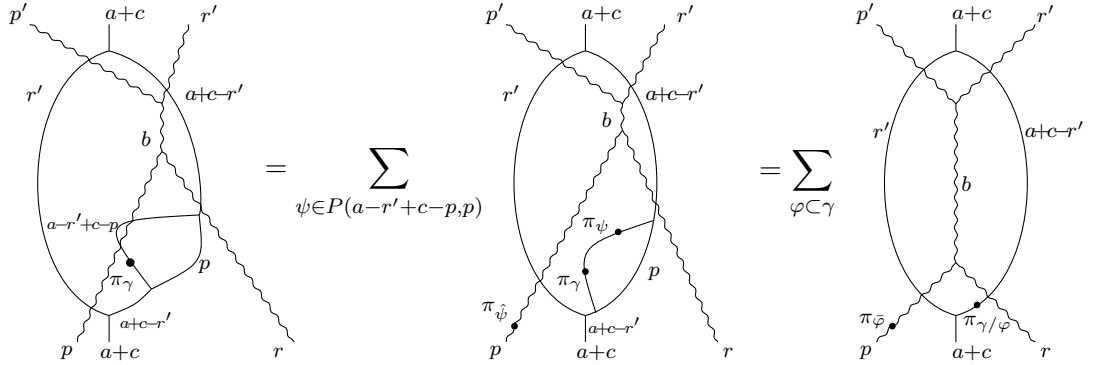
$$= \delta_{p,p'} \delta_{\alpha,\alpha'} (-1)^{|\alpha|+r(a+c-r)}$$

We shall prove the formula by simplifying the diagram on the left hand side:



where, by Lemma 2, $\gamma \in P(a-r'+c-p, r'-r)$ and $s_\gamma \in \{-1, 0, 1\}$. Thus, in order the last diagram to be nonzero, we must have $r' \geq r$. Moreover, if $r' = r$, by the second part of Lemma 2, we must also have $\alpha = \alpha'$ and $s_\gamma = (-1)^{|\alpha|}$.

Now, the last diagram (without the sign s_γ), by applying the Pitchfork lemma, Thick R2 move and Lemma 3, becomes:



On the last diagram, we shall apply Opening of a Thick Edge, for the curly line of thickness b . We have two possibilities: either $p \geq r'$ or $p < r'$. We shall assume that the first one is satisfied - the other case is done completely analogously.

So, let $p = r' + x$, for some $x \geq 0$. Note that then also $p' = r + x$. Then, the last diagram becomes:

$$\begin{aligned}
& \sum_{\varphi \subset \gamma} \text{Diagram 1} \stackrel{\text{Pitchfork}}{=} \sum_{\varphi \subset \gamma} \text{Diagram 2} \stackrel{\text{Thick R2}}{=} \\
& \sum_{\varphi \subset \gamma} \sum_{w \in P(r', x)} \text{Diagram 3} \stackrel{\text{Thick R3} + \text{Pitchfork}}{=} \\
& \sum_{\varphi \subset \gamma} \sum_{w \in P(r', x)} \sum_{i=0}^r \sum_{f_1, f_2, f_3 \in P(i, r'-i)} C_{f_1 f_2 f_3}^{K_i} \text{Diagram 4} \stackrel{\text{Thick R2}}{=} \\
& \sum_{\varphi \subset \gamma} \sum_{w \in P(r', x)} \sum_{i=0}^r \sum_{f_1, f_2, f_3 \in P(i, r'-i)} \sum_{y \in P(a+c-r', i)} C_{f_1 f_2 f_3}^{K_i} \text{Diagram 5}
\end{aligned}
\tag{24}$$

The diagrams are as follows:

- Diagram 1:** A central vertex with four wavy lines extending outwards. The top-left line is labeled p' , the top-right r' , the bottom-left p , and the bottom-right r . A vertical line segment labeled $a+c$ is on the left, and another labeled $a+c$ is on the right. A horizontal line segment labeled x is in the center. A point $\pi_{\bar{\varphi}}$ is on the left line, and a point $\pi_{\gamma/\varphi}$ is on the right line. A loop labeled r is on the left, and a loop labeled r' is on the right.
- Diagram 2:** Similar to Diagram 1, but with a different configuration of loops and points.
- Diagram 3:** Similar to Diagram 1, but with a different configuration of loops and points.
- Diagram 4:** Similar to Diagram 1, but with a different configuration of loops and points.
- Diagram 5:** Similar to Diagram 1, but with a different configuration of loops and points.

where $K_i = \underbrace{(r' - i, \dots, r' - i)}_i$, $i > 0$, and $K_i = 0$ for $i = 0$.

Although the last expression has many sums in it, only very few summands can be nonzero. First of all, we have that

$$\begin{array}{c} \pi_w \\ \bullet \\ \pi_{\bar{f}_3} \\ \bullet \\ | \\ r' \end{array} = \sum_{z \in P(r', x+i)} c_{w, \bar{f}_3}^z \begin{array}{c} | \\ \bullet \\ \pi_z \\ | \\ r' \end{array}$$

and

$$\begin{array}{c} \pi_{\gamma/\varphi} \\ \bullet \\ \pi_y \\ \bullet \\ | \\ a+c-r' \end{array} = \sum_{\nu \in P(a+c-r', r'-r)} c_{\varphi, \nu}^\gamma \begin{array}{c} \pi_\nu \\ \bullet \\ \pi_y \\ \bullet \\ | \\ a+c-r' \end{array} = \sum_{u \in P(a+c-r', r'-r+i)} \sum_{\nu \in P(a+c-r', r'-r)} c_{\varphi, \nu}^\gamma c_{y, \nu}^u \begin{array}{c} | \\ \bullet \\ \pi_u \\ | \\ a+c-r' \end{array}$$

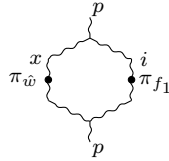
and so the digon on thick line can be written as:

$$\begin{array}{c} a+c \\ \swarrow \quad \searrow \\ \pi_{f_3} \quad \pi_y \\ \swarrow \quad \searrow \\ \pi_w \quad \pi_{\gamma/\varphi} \\ \swarrow \quad \searrow \\ a+c \end{array} = \sum_{z \in P(r', x+i)} \sum_{u \in P(a+c-r', r'-r+i)} \sum_{\nu \in P(a+c-r', r'-r)} c_{\varphi, \nu}^\gamma c_{y, \nu}^u c_{w, \bar{f}_3}^z \begin{array}{c} a+c \\ \swarrow \quad \searrow \\ \pi_z \quad \pi_u \\ \swarrow \quad \searrow \\ a+c \end{array} \quad (25)$$

Now, since $r' \geq r \geq i$ and by assumption $b \leq a+c$, we have that $x+i = p-r'+i \leq p-r'+r = b-r' \leq a+c-r'$ and $r'-r+i \leq r'$, and so $z \in P(r', a+c-r')$ and $u \in P(a+c-r', r')$. Thus, by Lemma 2, we have that the last diagram can be nonzero only when $|z| + |u| = r'(a+c-r')$, i.e. we must have

$$|w| + |f_3| + |y| + |\gamma| - |\varphi| = r'(a+c-r'). \quad (26)$$

As for the digon on curly lines:



again by Lemma 2 it can be nonzero only when $|\hat{w}| + |f_1| \geq xi$, i.e.:

$$r'x - |w| + |f_1| \geq xi. \quad (27)$$

Thus, from (26), (27), $|f_1| + |f_3| \leq |f_1| + |f_2| + |f_3| = i(r'-i)$ and since $y \in P(a+c-r', i)$, $\gamma \in P(a+c-r'-p, r'-r)$ and $x = p-r'$, we obtain:

$$r'(a+c-r') \leq (p-r')(r'-i) + i(r'-i) + (a+c-r')i + (a+c-r'-p)(r'-r) - |\varphi|.$$

The last can be rewritten as

$$|\varphi| + (a + c - i - p)(r - i) + (r' - i)(r' - r) \leq 0.$$

Since $r' \geq r \geq i$ and $a + c \geq b = p + r \geq p + i$, all terms on the left hand side must be equal to zero, i.e. we must have $r' = r = i$ and $\varphi = 0$. Moreover, since $r' = r$, we also have $\gamma = 0$, and so by Lemma 2 we have that $\alpha' = \alpha$ and $s_\gamma = (-1)^{|\alpha|}$. By replacing all this in (24), it becomes:

$$\delta_{r,r'} \delta_{\alpha,\alpha'} (-1)^{|\alpha|} \sum_{w \in P(r,x)} \sum_{y \in P(a+c-r,r)} \text{Diagram 1} \text{Diagram 2} \text{Diagram 3}$$

The diagrams are:
 Diagram 1: A hexagon with vertices labeled x (top-left), w (bottom-left), p (bottom), r (bottom-right), r (top-right), and p (top).
 Diagram 2: A diamond shape with vertices labeled w (left), r (bottom), y (right), and w (top). The top and bottom vertices are also labeled $a+c$.
 Diagram 3: A vertical line with a dot labeled y and a wavy line extending upwards and downwards, both labeled r .

Again, by Lemma 2, the last is nonzero only when $w = 0$ and $y = \underbrace{(r, \dots, r)}_{a+c-r}$ and thus (24) reduces to

$$\delta_{r,r'} \delta_{\alpha,\alpha'} (-1)^{|\alpha|} (-1)^{r(a+c-r)} \text{Diagram 4}$$

Diagram 4: Three vertical lines. The leftmost is wavy and labeled p at the bottom. The middle is straight and labeled $a+c$ at the bottom. The rightmost is wavy and labeled r at the bottom.

as wanted. ■

5 Categorification of the higher quantum Serre relations

In this section we give a direct categorification of the higher quantum Serre relations for type A . This will be done in the homotopy category of $\dot{\mathcal{U}}_3^+$. The higher quantum Serre relations:

$$\sum_{i=0}^a (-1)^i q^{\pm(a-b-1)i} E_1^{(a-i)} E_2^{(b)} E_1^{(i)} = 0, \quad a > b > 0, \quad (28)$$

state that certain alternating sum of monomials in $U_q^+(\mathfrak{sl}_3)$ is equal to zero. Thus, it is natural to categorify it by building a complex of objects of $\dot{\mathcal{U}}_3^+$ lifting those monomials which is homotopic to zero.

For a categorification of the relation (8) with the plus sign, our goal is to define a complex of the form

$$\begin{aligned} 0 \rightarrow \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \rightarrow \mathcal{E}_1^{(a-1)} \mathcal{E}_2^{(b)} \mathcal{E}_1\{a-b-1\} \rightarrow \mathcal{E}_1^{(a-2)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(2)}\{2(a-b-1)\} \rightarrow \dots \\ \dots \rightarrow \mathcal{E}_1 \mathcal{E}_2^{(b)} \mathcal{E}_1^{(a-1)}\{(a-1)(a-b-1)\} \rightarrow \mathcal{E}_2^{(b)} \mathcal{E}_1^{(a)}\{a(a-b-1)\} \rightarrow 0 \end{aligned} \quad (29)$$

that is homotopic to zero.

Theorem 6 *The following complex*

$$0 \longrightarrow \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \xrightarrow{\text{diagram}} \mathcal{E}_1^{(a-1)} \mathcal{E}_2^{(b)} \mathcal{E}_1\{a-b-1\} \xrightarrow{\text{diagram}} \mathcal{E}_1^{(a-2)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(2)}\{2(a-b-1)\} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{E}_1 \mathcal{E}_2^{(b)} \mathcal{E}_1^{(a-1)}\{(a-1)(a-b-1)\} \xrightarrow{\text{diagram}} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(a)}\{a(a-b-1)\} \longrightarrow 0$$

is homotopic to zero.

Proof: Denote $C_i := \mathcal{E}_1^{(a-i)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(i)}\{i(a-b-1)\}$, and

$$d_i := \text{diagram} : C_i \longrightarrow C_{i+1}, \quad i = 0, \dots, a-1.$$

Thus we have to show that the complex

$$\mathcal{C} : 0 \rightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{a-1}} C_a \rightarrow 0,$$

is homotopic to zero.

First of all, \mathcal{C} is indeed a complex, since

$$d_{i+1}d_i = \text{diagram} = \text{diagram} = 0.$$

In order to prove that \mathcal{C} is homotopic to zero, we are left with defining morphisms $h_i : C_{i+1} \rightarrow C_i$, for $i = 0, \dots, a-1$, such that

$$h_i d_i + d_{i-1} h_{i-1} = \text{Id}_{C_i}, \quad i = 1, \dots, a-1, \quad (30)$$

$$h_0 d_0 = \text{Id}_{C_0}, \quad (31)$$

$$d_{a-1} h_{a-1} = \text{Id}_{C_a}. \quad (32)$$

To that end, for every $i = 0, \dots, a-1$, we define $h_i : C_{i+1} \rightarrow C_i$, as follows:

$$h_i := (-1)^{a-1-i} \begin{array}{c} a-i \\ \diagup \quad \diagdown \\ a-b-1 \quad 1 \\ \diagdown \quad \diagup \\ a-(i+1) \quad b \quad i+1 \end{array}$$

Then, for every $i = 1, \dots, a-1$, we have

$$\begin{aligned} h_i d_i + d_{i-1} h_{i-1} &= (-1)^{a-1-i} \begin{array}{c} a-i \\ \diagup \quad \diagdown \\ a-b-1 \quad 1 \\ \diagdown \quad \diagup \\ a-i \quad b \quad i \end{array} + (-1)^{a-i} \begin{array}{c} a-i \\ \diagdown \quad \diagup \\ a-b-1 \quad 1 \\ \diagup \quad \diagdown \\ a-i \quad b \quad i \end{array} = \\ &= (-1)^{a-1-i} \begin{array}{c} a-i \\ \diagup \quad \diagdown \\ a-b-1 \quad 1 \\ \diagdown \quad \diagup \\ a-i \quad b \quad i \end{array} + (-1)^{a-i} \begin{array}{c} a-i \\ \diagdown \quad \diagup \\ 1 \quad a-b-1 \\ \diagup \quad \diagdown \\ a-i \quad b \quad i \end{array} = \\ &= (-1)^{a-1-i} \begin{array}{c} a-i \\ \diagup \quad \diagdown \\ a-b-1 \quad 1 \\ \diagdown \quad \diagup \\ a-i \quad b \quad i \end{array} + (-1)^{a-1-i} \sum_{x+y+z=b-1} \begin{array}{c} a-i \\ \diagup \quad \diagdown \\ 1 \quad a-b-1+x \\ \diagdown \quad \diagup \\ a-i \quad b \quad i \end{array} + \\ &\quad + (-1)^{a-i} \begin{array}{c} a-i \\ \diagdown \quad \diagup \\ 1 \quad a-b-1 \\ \diagup \quad \diagdown \\ a-i \quad b \quad i \end{array} \end{aligned} \tag{33}$$

Here in the second equality we have used Opening of a Thick Edge, while in the third equality we have applied the Thick R3 move on the first term. Now, by Dot Migration, the sum of the first and the third summand from above is equal to:

$$(-1)^{a-1-i} \sum_{r+s=a-b-2} \begin{array}{c} a-i \\ \diagup \quad \diagdown \\ r \quad s \\ \diagdown \quad \diagup \\ a-i \quad b \quad i \end{array} = (-1)^{a-1-i} \sum_{r+s=a-b-2} \sum_{x=0}^b \begin{array}{c} a-i \\ \diagup \quad \diagdown \\ r \quad x+s \\ \diagdown \quad \diagup \\ a-i \quad b \quad i \end{array} \tag{34}$$

where we have used the Thick R2 move.

The last double sum is nonzero only when $r \geq a-i-1$ and $x+s \geq i-1$. Since $x \leq b$ and $r+s = a-b-2$, the last implies that we must have

equalities, i.e. that $r = a - i - 1$, $x = b$ and $s = i - b - 1$. Finally, the last is possible only when $i \geq b + 1$, and so (34) is equal to $\delta_{\{i \geq b+1\}} \text{Id}_{C_i}$.

Analogously, the remaining second term from (33) is nonzero, only when $a - b - 1 + x \geq a - i - 1$, $y \geq i - 1$ and $z \geq 0$, and since $x + y + z = a - b - 2$, again we must have all equalities, and in that case the value of the second term is equal to Id_{C_i} . Since $x \geq 0$, we must have $i \leq b$, and so altogether we have:

$$h_i d_i + d_{i-1} h_{i-1} = \delta_{\{i \geq b+1\}} \text{Id}_{C_i} + \delta_{\{i \leq b\}} \text{Id}_{C_i} = \text{Id}_{C_i},$$

thus proving (30). The equalities (31) and (32) can be obtained completely analogously. Hence, \mathcal{C} is homotopic to zero, as wanted. ■

As for the categorification of the higher quantum Serre relations (28) with the minus sign, we obtain it by defining a complex of the form (29), but with the arrows pointing in the opposite direction. By exchanging the roles of d_i 's and h_i 's from the Theorem above, we have:

Theorem 7 *The following complex*

$$\begin{aligned} 0 \leftarrow \mathcal{E}_1^{(a)} \mathcal{E}_2^{(b)} \xleftarrow{h_1} \mathcal{E}_1^{(a-1)} \mathcal{E}_2^{(b)} \mathcal{E}_1\{-(a-b-1)\} \xleftarrow{h_2} \\ \mathcal{E}_1^{(a-2)} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(2)}\{-2(a-b-1)\} \xleftarrow{h_3} \dots \xleftarrow{h_{a-1}} \mathcal{E}_2^{(b)} \mathcal{E}_1^{(a)}\{-a(a-b-1)\} \leftarrow 0, \end{aligned}$$

is homotopic to zero.

References

- [1] J. Brundan, A. Kleshchev: *Graded decomposition numbers for cyclotomic Hecke algebras*, Advances in Math. 222 (2009), 1883-1942.
- [2] W. Fulton: *Young Tableaux: with Applications to Representation Theory and Geometry*, Cambridge University Press, Cambridge, 1997.
- [3] M. Khovanov, A. Lauda: *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory 13 (2009), 309-347.
- [4] M. Khovanov, A. Lauda: *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. 363 (2011), 2685-2700.
- [5] M. Khovanov, A. Lauda: *A diagrammatic approach to categorification of quantum groups III*, Quantum Topology, Vol 1, Issue 1 (2010), 1-92.
- [6] M. Khovanov, A. Lauda, M. Mackaay, M. Stošić: *Extended graphical calculus for categorified quantum $sl(2)$* , arXiv:1006.2866.
- [7] A. Lauda: *A categorification of quantum $sl(2)$* , Advances in Mathematics, Volume 225, Issue 6 (2010), 3327-3424.
- [8] G. Lusztig: *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*, Birkhäuser, Boston, 1993.
- [9] G. Lusztig: *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 3 (1990), 447-498.
- [10] M. Mackaay, M. Stošić, P. Vaz: *Extended graphical calculus for categorified quantum $sl(n)$* , in preparation.

- [11] M. Varagnolo, E. Vasserot: *Canonical basis and KLR-algebras*, arXiv:0901.3992.
- [12] B. Webster: *Knots and higher representation theory I: diagrammatic and geometric categorification of tensor products*, arXiv:1001.2020.
- [13] B. Webster: *Knots and higher representation theory II: the categorification of quantum knot invariants*, arXiv:1005.4559.

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